The complete problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 T}{\partial \varphi^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{k}g(r,\varphi,z,t)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

B.C.
$$\begin{cases} T|_{r=0}=\text{finite} \\ T|_{r=b}=0 \\ \begin{cases} T|_{z=0}=0 \\ T|_{z=L}=0 \end{cases} \\ T(\varphi+2\pi)=T(\varphi) \end{cases}$$

I.C. $T|_{t=0}=T_i(r,\varphi,z)$

Consider the auxiliary homogeneous problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 \psi}{\partial \varphi^2}+\frac{\partial^2 \psi}{\partial z^2}=\frac{1}{\alpha}\frac{\partial \psi}{\partial t}$$

B.C.
$$\begin{cases} \psi|_{r=0}=\text{finite} \\ \psi|_{r=b}=0 \\ \begin{cases} \psi|_{z=0}=0 \\ \psi|_{z=L}=0 \end{cases} \\ \psi(\varphi+2\pi)=\psi(\varphi) \end{cases}$$

I.C. $\psi|_{t=0}=T_i(r,\varphi,z)$

The general solution:

$$\psi(r,\varphi,z,t)=\sum_{l,m,n}(A\cos m\varphi+B\sin m\varphi)\sin\beta_n z\cdot J_m(\lambda_l r)e^{-(\lambda_l^2+\beta_n^2)\alpha t}$$

where:
$$\begin{cases} m = 0, 1, 2, 3, \ldots \\ \beta_n = \dfrac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \\ \lambda_\ell \text{ satisfying } J_m(\lambda_\ell b) = 0, \quad \ell = 1, 2, 3, \ldots \end{cases}$$

Using initial condition:

$$\Psi\big|_{t=0} = T_i(r, \varphi, z)$$

So:
$$T_i(r,\varphi,z) = \sum_{\ell, m, n} (A\cos m\varphi + B\sin m\varphi) \sin\beta_n z \, J_m(\lambda_\ell r)$$

$$\Psi(r,\varphi,z) = \sum_{\ell,m,n} \left[\frac{\int_0^L\int_0^{2\pi}\int_0^b T_i(r',\varphi',z') \cos m\varphi' \sin\beta_n z' J_m(\lambda_\ell r') r' dr' d\varphi' dz'}{\int_0^{2\pi} \cos^2 m\varphi \, d\varphi \int_0^L \sin^2\beta_n z \, dz \int_0^b J_m^2(\lambda_\ell r) r dr} \times \right.$$

$$\times \cos m\varphi \sin\beta_n z \, J_m(\lambda_\ell r) + \frac{\int_0^L\int_0^{2\pi}\int_0^b T_i(r',\varphi',z') \sin m\varphi' \sin\beta_n z' J_m(\lambda_\ell r') r' dr' d\varphi' dz'}{\int_0^{2\pi} \sin^2 m\varphi \int_0^L \sin^2\beta_n z \, dz \int_0^b J_m^2(\lambda_\ell r) r dr} \times$$

$$\left. \times \sin m\varphi \sin\beta_n z \, J_m(\lambda_\ell r) \right]$$

i.e.:
$$\Psi(r,\varphi,z) = \int_0^L\int_0^{2\pi}\int_0^b \sum_{\ell m n} \frac{(\cos m\varphi' \cos m\varphi + \sin m\varphi' \sin m\varphi) \sin\beta_n z' \sin\beta_n z \, J_m(\lambda_\ell r') J_m(\lambda_\ell r) e^{-\alpha(\lambda_\ell^2 + \beta_n^2)t}}{\frac{\pi L}{2} \int_0^b J_m^2(\lambda_\ell r) r dr} \times$$

$$\times T_i(r', \varphi', z') r' dr' d\varphi' dz'$$

So: $$G(r,\varphi,z,t|r',\varphi',z',0)=\sum_{\ell mn}\frac{e^{-\alpha(\lambda_\ell^2+\beta_n^2)t}\cos(\varphi-\varphi')\sin\beta_n z'\sin\beta_n z\,J_m(\lambda_\ell r')J_m(\lambda_\ell r)}{\frac{\pi L}{2}\int_0^b J_m^2(\lambda_\ell r)\,r\,dr}$$

$$G(r,\varphi,z,t|r',\varphi',z',\tau)=\sum_{\ell mn}\frac{e^{-\alpha(\lambda_\ell^2+\beta_n^2)(t-\tau)}\cos(\varphi-\varphi')\sin\beta_n z'\sin\beta_n z\,J_m(\lambda_\ell r')J_m(\lambda_\ell r)}{\frac{\pi L}{2}\int_0^b J_m^2(\lambda_\ell r)\,r\,dr}$$

and: $$T(r,\varphi,z,t)=\int_0^L\int_0^{2\pi}\int_0^b G(r,\varphi,z,t|r',\varphi',z',0)\,T_i(r',\varphi',z')\,r'dr'd\varphi'dz'$$

$$+\frac{\alpha}{K}\int_0^t\int_0^L\int_0^{2\pi}\int_0^b G(r,\varphi,z,t|r',\varphi',z',\tau)\,g(r',\varphi',z',\tau)\,r'dr'd\varphi'dz'd\tau$$