The complete problem: $T(r,\xi,t)$

$$\boxed{\begin{aligned}
&\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial T}{\partial \xi}\right]=\frac{1}{\alpha}\frac{\partial T}{\partial t}\\
&\underline{\text{B.C.}}\quad \begin{cases} T\big|_{r=0}=\text{finite}\\ \left.\dfrac{\partial T}{\partial r}\right|_{r=b}=0\end{cases}\\
&\qquad\quad \begin{cases} T\big|_{\xi=\pm 1}=\text{finite}\end{cases}\\
&\underline{\text{I.C.}}\quad T\big|_{t=0}=F(r,\xi)
\end{aligned}}$$

Define a new dependent variable $V(r,\xi,t)\equiv r^{\frac{1}{2}}\,T(r,\xi,t)$

therefore:

$$\boxed{\begin{aligned}
&\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}-\frac{1}{4}\frac{V}{r^2}+\frac{1}{r^2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial V}{\partial \xi}\right]=\frac{1}{\alpha}\frac{\partial V}{\partial t}\\
&\underline{\text{B.C.}}\quad \begin{cases} V\big|_{r=0}=0\\ \left.\dfrac{\partial V}{\partial r}\right|_{r=b}-\dfrac{1}{2b}V\Big|_{r=b}=0\end{cases}\\
&\qquad\quad \begin{cases} V\big|_{\xi=\pm 1}=\text{finite}\end{cases}\\
&\underline{\text{I.C.}}\quad V\big|_{t=0}=r^{\frac{1}{2}}F(r,\xi)
\end{aligned}}$$

① Let: $V(r,\xi,t)=R(r)\,\Theta(\xi)\,\Gamma(t)$

then: $$\frac{R''+\frac{1}{r}R'-\frac{1}{4r^2}R}{R}+\frac{\frac{1}{r^2}\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right]}{\Theta}=\underbrace{\frac{\frac{1}{\alpha}\frac{d\Gamma}{dt}}{\Gamma}=-\lambda^2}_{\Gamma(t)=Ee^{-\lambda^2\alpha t}}$$

For R and Θ:

$$\frac{R''+\frac{1}{r}R'-\frac{1}{4r^2}R}{R}+\lambda^2=-\frac{1}{r^2}\frac{\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right]}{\Theta}$$

so. $$\frac{r^2R''+rR'-\frac{1}{4}R+\lambda^2r^2R}{R}=-\frac{\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right]}{\Theta}=\mu$$

Then: $$\begin{cases}\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right]+\mu\Theta=0\\ r^2R''+rR'+\left[\lambda^2r^2-\frac{1}{4}-\mu\right]R=0\end{cases}$$

(2). For $\Theta(\xi)$:

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right]+\mu\Theta=0$$

B.C. $\Theta\big|_{\xi=\pm1}=$ finite

general solution: $\boxed{\Theta(\xi)=CP_n(\xi)}$ $n=0,1,2,\cdots$

with $\mu=n(n+1)$

For $R(r)$:

$$r^2R''+rR'+\left[\lambda^2r^2-\frac{1}{4}-n(n+1)\right]R=0$$

$$R''+\frac{1}{r}R'+\left[\lambda^2-(n+\tfrac{1}{2})^2\frac{1}{r^2}\right]R=0$$

general solution: $R(r)=AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r)$

B.C.: $R\big|_{r=0}=0$ $\Rightarrow$ $B=0$ $\left(Y_{n+\frac{1}{2}}(0)\to\infty\right)$

$(V|_{r=0}=0)$

and: $R(r)=AJ_{n+\frac{1}{2}}(\lambda r)$

B.C. $\left.\frac{\partial V}{\partial r}\right|_{r=b} - \frac{1}{2b} V\Big|_{r=b} = 0$

so: $\left.\frac{dR}{dr}\right|_{r=b} - \frac{1}{2b} R\Big|_{r=b} = 0$

i.e.: $\lambda J'_{n+\frac{1}{2}}(\lambda b) - \frac{1}{2b} J_{n+\frac{1}{2}}(\lambda b) = 0$

$$\boxed{2b\lambda = \frac{J_{n+\frac{1}{2}}(\lambda b)}{J'_{n+\frac{1}{2}}(\lambda b)}}$$ (for each n) eigenvalue problem!

with $\lambda = \lambda_{nm}$ with $m = 1, 2, \ldots$

therefore, for each n and m:

$$\boxed{R(r) = A J_{n+1/2}(\lambda_{nm} r)}$$

(3) For each n and m:

$$V_{nm}(r, \xi, t) = C_{nm} J_{n+\frac{1}{2}}(\lambda_{nm} r) P_n(\xi) e^{-\lambda_{nm}^2 \alpha t}$$

$$V(r, \xi, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} J_{n+\frac{1}{2}}(\lambda_{nm} r) P_n(\xi) e^{-\lambda_{nm}^2 \alpha t}$$

(4) Applying initial condition:

$$V\Big|_{t=0} = r^{\frac{1}{2}} F(r, \xi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} J_{n+\frac{1}{2}}(\lambda_{nm} r) P_n(\xi)$$

$$C_{nm} = \frac{\int_{r=0}^{b} \int_{\xi=-1}^{+1} r^{3/2} J_{n+\frac{1}{2}}(\lambda_{nm} r) P_n(\xi) F(r, \xi) d\xi \, dr}{\int_{\xi=-1}^{1} [P_n(\xi)]^2 d\xi \cdot \int_{r=0}^{b} J_{n+\frac{1}{2}}^2(\lambda_{nm} r) r \, dr}$$