The complete problem: $T(r,t)$

$$\boxed{\begin{aligned} &\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \\ &\text{B.C.}\ \begin{cases} T\big|_{r=0} = \text{finite} \\ -k\frac{\partial T}{\partial r}\Big|_{r=b} = h\left(T\big|_{r=b} - T_\infty\right) \end{cases} \\ &\text{I.C.}\ T\big|_{t=0} = T_i(r) \end{aligned}}$$

Define: $\theta(r,t) \equiv T(r,t) - T_\infty$

$$\boxed{\begin{aligned} &\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \\ &\text{B.C.}\ \begin{cases} \theta\big|_{r=0} = \text{finite} \\ -k\frac{\partial \theta}{\partial r}\Big|_{r=b} = h\theta\big|_{r=b} \end{cases} \\ &\text{I.C.}\ \theta\big|_{t=0} = T_i(r) - T_\infty \equiv \theta_i(r) \end{aligned}}$$

① Let: $\theta(r,t) = R(r)\Gamma(t)$

so:
$$\frac{R'' + \frac{1}{r}R'}{R} = \underbrace{\frac{\frac{1}{\alpha}\frac{d\Gamma}{dt}}{\Gamma} = -\lambda^2}$$

$$\Downarrow$$

$$\Gamma = Ae^{-\lambda^2 \alpha t}$$

and
$$\frac{R'' + \frac{1}{r}R'}{R} = -\lambda^2$$

i.e.: $R'' + \frac{1}{r}R' + \lambda^2 R = 0$

② For $R(r)$: $\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda^2 R = 0$

General solution: $R(r) = B J_0(\lambda r) + C Y_0(\lambda r)$

B.C. $\theta\big|_{r=0} = \text{finite} \Rightarrow R\big|_{r=0} = \text{finite} \Rightarrow C = 0$

(because $Y_0(0) \to -\infty$)

So: $R(r) = B J_0(\lambda r)$

B.C. $-k\frac{\partial\theta}{\partial r}\Big|_{r=b} = h\theta\big|_{r=b} \Rightarrow -k\frac{dR}{dr}\Big|_{r=b} = hR\big|_{r=b}$

$\Rightarrow -k\lambda B J_0'(\lambda b) = hBJ_0(\lambda b)$

B cannot be 0, otherwise trivial sol^n!

So: $-\frac{k\lambda}{h} = \frac{J_0(\lambda b)}{J_0'(\lambda b)}$

i.e.: $\boxed{\lambda\frac{k}{h} = \frac{J_0(\lambda b)}{J_1(\lambda b)}}$ the eigenvalue problem!

$\lambda = \lambda_n, \; n = 1, 2, 3, \ldots$

③ $\theta(r,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \alpha t} J_0(\lambda_n r)$

④ Applying I.C.:

$$\theta\big|_{t=0} = \theta_i(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

Using orthogonal property: $\int_0^b J_0(\lambda_n r) J_0(\lambda_m r)\, r\,dr = 0$ for $m \neq n$.

so:
$$A_n = \frac{\int_0^b \theta_i(r) J_0(\lambda_n r)\, r\, dr}{\int_0^b J_0^2(\lambda_n r)\, r\, dr}$$

$$= \frac{2\int_0^b \theta_i(r) J_0(\lambda_n r)\, r\, dr}{b^2\left[J_0^2(\lambda_n b) + J_1^2(\lambda_n b)\right]}$$

therefore:
$$T(r,t) = \theta(r,t) + T_\infty$$

$$= T_\infty + \sum_{n=1}^{\infty} \frac{2\int_0^b \theta_i(r') J_0(\lambda_n r')\, r'\, dr'}{b^2\left[J_0^2(\lambda_n b) + J_1^2(\lambda_n b)\right]} \cdot e^{-\lambda_n^2 \alpha t} J_0(\lambda_n r)$$