The complete problem.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}+\frac{1}{k}g(z)=0 \qquad \left(g(z)=g_0 e^{-\alpha z} \right)$$

B.C.
$$\begin{cases} T\big|_{r=R_0}=T_0 \\ T\big|_{r=0}=\text{finite} \end{cases}$$

$$\begin{cases} T\big|_{z=L}=T_0 \\ \left.\frac{\partial T}{\partial z}\right|_{z=0}=0 \end{cases}$$

First: find a particular solution $T_p(z)$ satisfying:

$$\boxed{\begin{array}{l} \dfrac{\partial^2 T_p}{\partial z^2}+\dfrac{1}{k}g_0 e^{-\alpha z}=0 \\ \text{B.C.} \quad \begin{cases} T_p\big|_{z=L}=T_0 \\ \left.\dfrac{dT_p}{dz}\right|_{z=0}=0 \end{cases} \end{array}}$$

$$T_p(z)=-\frac{g_0}{\alpha^2 k}e^{-\alpha z}+C_1 z+C_2$$

from B.C.: $\begin{cases} C_1=-\dfrac{g_0}{\alpha k} \\ C_2=\dfrac{g_0}{\alpha k}\left(\dfrac{e^{-\alpha L}}{\alpha}+L\right)+T_0 \end{cases}$

Therefore: $$T_p(z)=T_0-\frac{g_0}{\alpha^2 k}\left(e^{-\alpha z}-e^{-\alpha L}\right)-\frac{g_0}{\alpha k}(z-L)$$

Let: $T(r,z) \equiv T_I(r,z) + T_p(z)$

So:

$$\boxed{\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_I}{\partial r}\right) + \frac{\partial^2 T_I}{\partial z^2} = 0 \\ &\text{B.C.} \begin{cases} T_I\big|_{r=0} = \text{finite} \\ T_I\big|_{r=R_0} = T_0 - T_p(z) = \frac{\delta_0}{\alpha^2 k}\left(e^{-\alpha z} - e^{-\alpha L}\right) + \frac{\delta_0}{\alpha k}(z-L) \\ T_I\big|_{z=L} = 0 \leftarrow (T_0 - T_p\big|_{z=L}) \\ \frac{\partial T_I}{\partial z}\Big|_{z=0} = 0 \end{cases} \end{aligned}}$$

with only one Nonhomogeneous B.C.!

Step ①

Assume: $T_I(r,z) = R(r)\,Z(z)$

then:

$$\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = \mu$$

So:

$$\begin{cases} R'' + \frac{1}{r}R' - \mu R = 0 \\ Z'' + \mu Z = 0 \end{cases}$$

step ②

According to homogeneous B.C.s for $Z(z)$, we must have: $\mu > 0$

Let $\mu \equiv \lambda^2$

For $Z(z)$, so $Z'' + \lambda^2 Z = 0$

general solution: $Z = A\sin\lambda z + B\cos\lambda z$.

imposing B.C. $\left.\frac{\partial T_I}{\partial z}\right|_{z=0} = 0 \Rightarrow \left.\frac{dZ}{dz}\right|_{z=0} = 0 \Rightarrow A = 0$

then: $Z(z) = B\cos\lambda z$

imposing B.C. $T_I\big|_{z=L} = 0 \Rightarrow Z\big|_{z=L} = 0$

so $B\cos\lambda L = 0$ (B cannot be 0)

therefore: $\cos\lambda L = 0$

$$\boxed{\lambda_n = \frac{(n+\frac{1}{2})\pi}{L}} \qquad n = 0, 1, 2, \cdots$$

eigenvalues

and for each n: $Z_n(z) = B\cos\frac{(n+\frac{1}{2})\pi}{L}z$

For $R(r)$:

$$R'' + \frac{1}{r}R' - \lambda_n^2 R = 0$$

general solution: $R(r) = C I_0(\lambda_n r) + D K_0(\lambda_n r)$

imposing B.C. $T_I\big|_{r=0} = \text{finite} \Rightarrow R\big|_{r=0} = \text{finite} \Rightarrow D = 0$

so: $R_n(r) = C I_0(\lambda_n r)$

Step 3

Therefore, for each $n = 0, 1, 2 \cdots$

$$T_{I,n}(r,z) = C_n I_0(\lambda_n r)\cos\left(\frac{n+\frac{1}{2}\pi}{L}z\right) \qquad \left(\lambda_n = \frac{(n+\frac{1}{2})\pi}{L}\right)$$

$(\cos\lambda_n z)$

Then: $T_I(r,z) = \sum_{n=0}^{\infty} C_n I_0(\lambda_n r)\cos\lambda_n z$

Step (4)

Applying Nonhomogeneous B.C. at $T_I\Big|_{r=R_0}$.

$$\frac{g_0}{\alpha^2 k}\left(e^{-\alpha z} - e^{-\alpha L}\right) + \frac{g_0}{\alpha k}(z-L) = \sum_{n=0}^{\infty} C_n I_0(\lambda_n R_0)\cos\lambda_n z$$

So: $$C_n I_0(\lambda_n R_0) = \frac{2}{L}\int_0^L \left[\frac{g_0}{\alpha^2 k}\left(e^{-\alpha z} - e^{-\alpha L}\right) + \frac{g_0}{\alpha k}(z-L)\right]\cos\lambda_n z\, dz$$

i.e. $$C_n = \frac{\frac{2}{L}\int_0^L \left[\frac{g_0}{\alpha^2 k}\left(e^{-\alpha z} - e^{-\alpha L}\right) + \frac{g_0}{\alpha k}(z-L)\right]\cos\frac{(n+\frac{1}{2})\pi z}{L}\, dz}{I_0\left(\frac{(n+\frac{1}{2})\pi R_0}{L}\right)}$$

So we have: $T(r,z) = T_I(r,z) + T_p(z)$

Note: $$\frac{2}{L}\int_0^L \left[\frac{g_0}{\alpha^2 k}\left(e^{-\alpha z} - e^{-\alpha L}\right) + \frac{g_0}{\alpha k}(z-L)\right]\cos\frac{(n+\frac{1}{2})\pi z}{L}\, dz$$
$$= \frac{2 g_0}{L k}\,\frac{\alpha + (-1)^n \frac{(n+\frac{1}{2})\pi}{L} e^{-\alpha L}}{\left(\alpha^2 + \frac{(n+\frac{1}{2})^2\pi^2}{L^2}\right)\cdot\frac{(n+\frac{1}{2})^2\pi^2}{L^2}}$$